Rotating a 3D Vector to the X-Axis

1 Problem Statement

Given a vector $\mathbf{v} = (v_x, v_y, v_z)$ in 3D space at the origin, find the two rotation angles needed to:

1. Rotate about the x-axis to place the vector in the xz-plane (eliminate v_y)
2. Rotate about the y-axis to align with the positive x-axis (eliminate v_z)

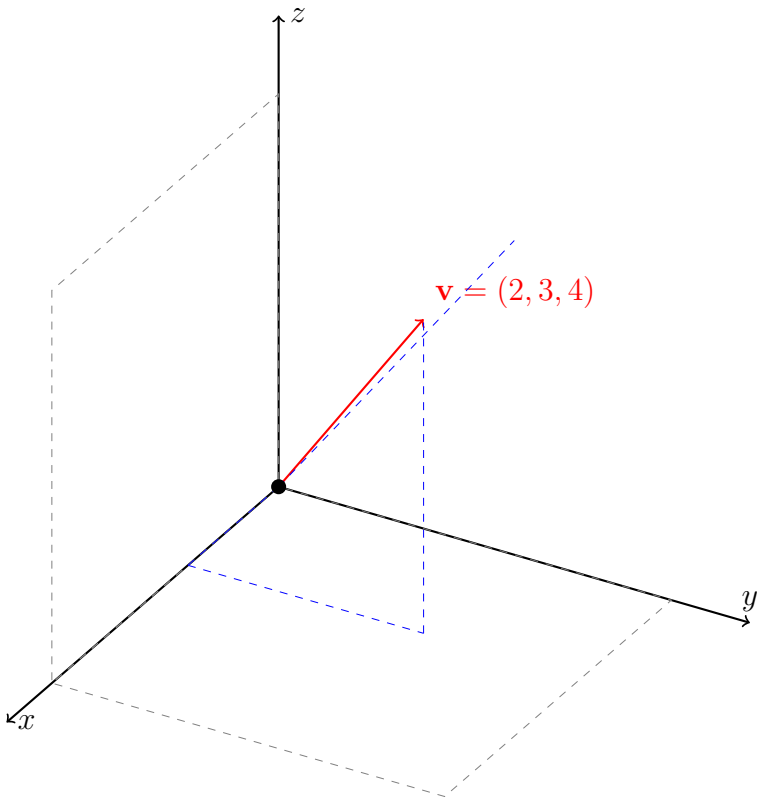
Example: Transform the vector $\mathbf{v} = (2, 3, 4)$ to become collinear with the x-axis.

2 Initial Configuration

2.1 Vector Components and Magnitude

$$v_x = 2 \tag{1}$$
$$v_y = 3 \tag{2}$$
$$v_z = 4 \tag{3}$$
$$|\mathbf{v}| = \sqrt{v_x^2 + v_y^2 + v_z^2} = \sqrt{4 + 9 + 16} = \sqrt{29} \approx 5.385 \tag{4}$$

2.2 3D Visualization of Initial Vector



3 Step 1: Rotate About X-Axis to Eliminate y-Component

3.1 Objective

Rotate the vector about the x-axis so that $v_y = 0$, placing the entire vector in the xz-plane.

3.2 Trigonometric Analysis

When looking along the x-axis (from the positive x toward the origin), the yz-plane appears as a 2D coordinate system. The vector's projection onto this plane is $(v_y, v_z) = (3, 4)$.

We need to find the angle to rotate this 2D vector to the form $(0, r_1)$:

- y-component: $v_y = 3$
- z-component: $v_z = 4$
- Hypotenuse in yz-plane: $r_1 = \sqrt{v_y^2 + v_z^2} = \sqrt{9 + 16} = \sqrt{25} = 5$

Rotation angle about the x-axis:

$$\theta_1 = \arctan 2(v_y, v_z) = \arctan 2(3, 4) \tag{5}$$

$$\sin(\theta_1) = \frac{v_y}{r_1} = \frac{3}{5} = 0.6 \tag{6}$$

$$\cos(\theta_1) = \frac{v_z}{r_1} = \frac{4}{5} = 0.8 \tag{7}$$

$$\theta_1 \approx 36.87^\circ \tag{8}$$

3.3 Rotation Matrix About X-Axis

$$R_X(\theta_1) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta_1) & -\sin(\theta_1) \\ 0 & \sin(\theta_1) & \cos(\theta_1) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0.8 & -0.6 \\ 0 & 0.6 & 0.8 \end{pmatrix} \tag{9}$$

3.4 Apply Rotation to Get Intermediate Vector

$$\mathbf{v}^{(1)} = R_X(\theta_1) \cdot \mathbf{v} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0.8 & -0.6 \\ 0 & 0.6 & 0.8 \end{pmatrix} \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} \tag{10}$$

$$v_x^{(1)} = 1(2) + 0(3) + 0(4) = 2 \tag{11}$$

$$v_y^{(1)} = 0(2) + 0.8(3) + (-0.6)(4) = 2.4 - 2.4 = 0 \quad \checkmark \tag{12}$$

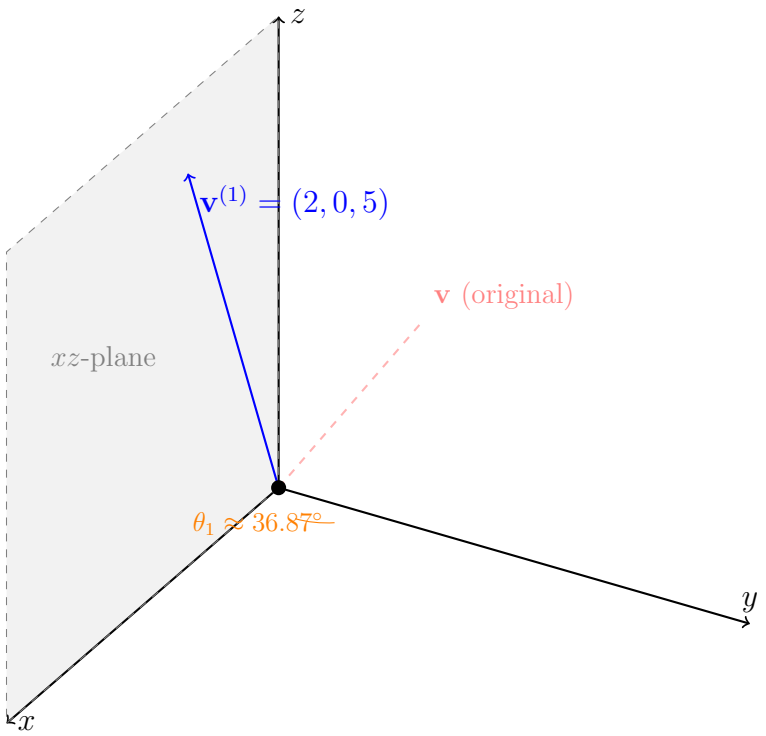
$$v_z^{(1)} = 0(2) + 0.6(3) + 0.8(4) = 1.8 + 3.2 = 5 \tag{13}$$

Result: $\mathbf{v}^{(1)} = (2, 0, 5)$ is now in the xz-plane!

3.5 Verify Magnitude is Preserved

$$|\mathbf{v}^{(1)}| = \sqrt{2^2 + 0^2 + 5^2} = \sqrt{4 + 25} = \sqrt{29} = |\mathbf{v}| \quad \checkmark \tag{14}$$

4 Intermediate State Visualization



5 Step 2: Rotate About Y-Axis to Eliminate z-Component

5.1 Objective

Rotate the intermediate vector about the y-axis so that $v_z = 0$, aligning the vector with the positive x-axis.

5.2 Trigonometric Analysis

Now working with the xz-plane projection: $(v_x, v_z) = (2, 5)$ from $\mathbf{v}^{(1)}$.

We need to rotate this 2D vector to the form $(\ell, 0)$ where $\ell = |\mathbf{v}|$:

- x-component: $v_x = 2$
- z-component: $v_z = 5$
- Hypotenuse in xz-plane: $\ell = \sqrt{v_x^2 + v_z^2} = \sqrt{4 + 25} = \sqrt{29} \approx 5.385$

Rotation angle about the y-axis:

$$\theta_2 = \arctan 2(v_z, v_x) = \arctan 2(5, 2) \tag{15}$$

$$\sin(\theta_2) = \frac{v_z}{\ell} = \frac{5}{\sqrt{29}} \approx 0.9285 \tag{16}$$

$$\cos(\theta_2) = \frac{v_x}{\ell} = \frac{2}{\sqrt{29}} \approx 0.3714 \tag{17}$$

$$\theta_2 \approx 68.20^\circ \tag{18}$$

5.3 Rotation Matrix About Y-Axis

$$R_Y(\theta_2) = \begin{pmatrix} \cos(\theta_2) & 0 & \sin(\theta_2) \\ 0 & 1 & 0 \\ -\sin(\theta_2) & 0 & \cos(\theta_2) \end{pmatrix} = \begin{pmatrix} 0.3714 & 0 & 0.9285 \\ 0 & 1 & 0 \\ -0.9285 & 0 & 0.3714 \end{pmatrix} \tag{19}$$

5.4 Apply Rotation to Get Final Vector

$$\mathbf{v}^{(2)} = R_Y(\theta_2) \cdot \mathbf{v}^{(1)} = \begin{pmatrix} 0.3714 & 0 & 0.9285 \\ 0 & 1 & 0 \\ -0.9285 & 0 & 0.3714 \end{pmatrix} \begin{pmatrix} 2 \\ 0 \\ 5 \end{pmatrix} \tag{20}$$

$$v_x^{(2)} = 0.3714(2) + 0(0) + 0.9285(5) = 0.7428 + 4.6425 = 5.385 \tag{21}$$

$$v_y^{(2)} = 0(2) + 1(0) + 0(5) = 0 \tag{22}$$

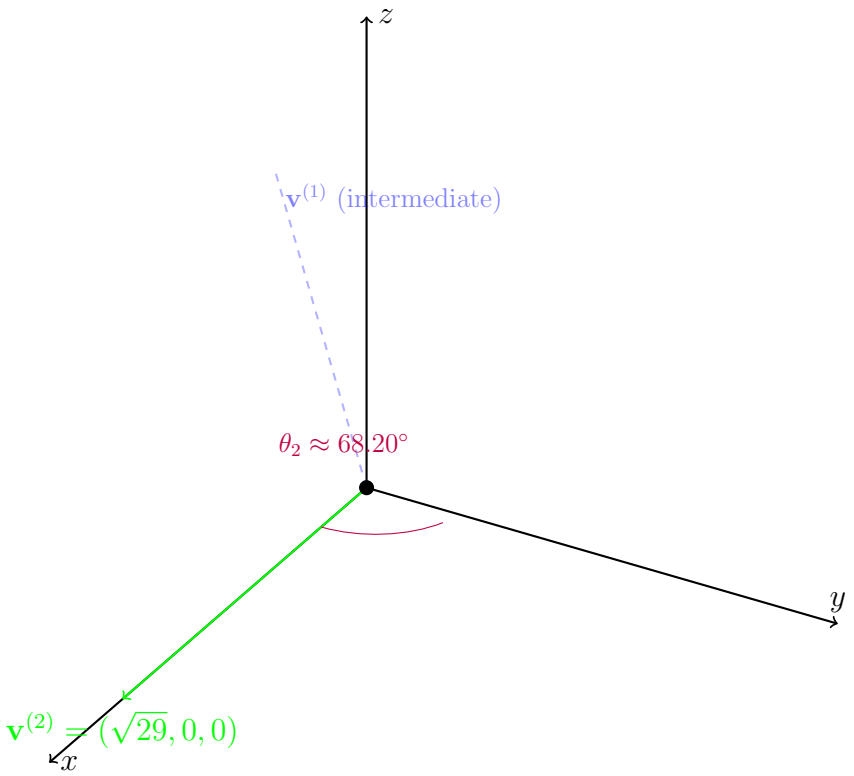
$$v_z^{(2)} = -0.9285(2) + 0(0) + 0.3714(5) = -1.857 + 1.857 = 0 \quad \checkmark \tag{23}$$

Result: $\mathbf{v}^{(2)} = (5.385, 0, 0) = (\sqrt{29}, 0, 0)$ is now aligned with the x-axis!

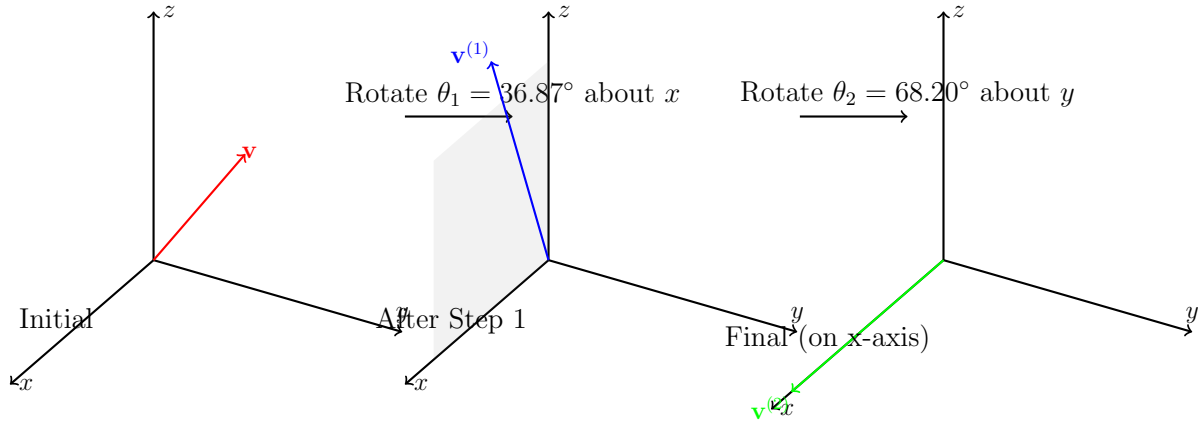
5.5 Verify Magnitude is Preserved

$$|\mathbf{v}^{(2)}| = \sqrt{5.385^2 + 0^2 + 0^2} = 5.385 = \sqrt{29} = |\mathbf{v}| \quad \checkmark \tag{24}$$

6 Final State Visualization



7 Complete Transformation Overview



8 Summary Table

Stage	Vector	Magnitude	Notes
Initial	$(2, 3, 4)$	$\sqrt{29} \approx 5.385$	3D vector
After θ_1	$(2, 0, 5)$	$\sqrt{29} \approx 5.385$	In xz-plane
Final	$(5.385, 0, 0)$	$\sqrt{29} \approx 5.385$	On x-axis

9 Key Rotation Angles

Step	Rotation	Angle	Purpose
1	About x-axis	$\theta_1 = \arctan 2(3,4) \approx 36.87°$	Eliminate v_y
2	About y-axis	$\theta_2 = \arctan 2(5,2) \approx 68.20°$	Eliminate v_z

10 General Formulas

For any vector $\mathbf{v} = (v_x, v_y, v_z)$:

Step 1 — Rotation about x-axis to reach xz-plane:

$$\theta_1 = \arctan 2(v_y, v_z) \tag{25}$$

After this rotation: $\mathbf{v}^{(1)} = (v_x, 0, r_1)$ where $r_1 = \sqrt{v_y^2 + v_z^2}$

Step 2 — Rotation about y-axis to reach x-axis:

$$\theta_2 = \arctan 2(v_z^{(1)}, v_x^{(1)}) = \arctan 2(r_1, v_x) \tag{26}$$

After this rotation: $\mathbf{v}^{(2)} = (|\mathbf{v}|, 0, 0)$

11 Combined Rotation Matrix

The complete transformation combining both rotations:

$$M = R_Y(\theta_2) \cdot R_X(\theta_1) \tag{27}$$

Applied to the original vector:

$$\mathbf{v}^{(2)} = M \cdot \mathbf{v} = R_Y(\theta_2) \cdot R_X(\theta_1) \cdot \mathbf{v} \tag{28}$$

$$M = \begin{pmatrix} 0.3714 & 0 & 0.9285 \\ 0 & 1 & 0 \\ -0.9285 & 0 & 0.3714 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0.8 & -0.6 \\ 0 & 0.6 & 0.8 \end{pmatrix} \tag{29}$$

$$M = \begin{pmatrix} 0.3714 & 0.5571 & 0.7428 \\ 0 & 0.8 & -0.6 \\ -0.9285 & 0.2229 & 0.2971 \end{pmatrix} \tag{30}$$

12 Key Takeaways

- **Two sequential rotations** transform any 3D vector to the x-axis
- **First rotation** about x-axis eliminates the y-component

- **Second rotation** about y-axis eliminates the z-component
- **Magnitude is preserved** through both rotations
- **Angles are computed** using $\arctan 2$ from 2D projections
- Each rotation can be understood as a simple 2D rotation in the appropriate plane